This week

1. Section 7.2: separable differential equations
2. Section 9.4: autonomous differential equations
A separable differential equation is an equation of the form
\[
\frac{dy}{dx} = f(x)g(y)
\]

Solution method:
1. Separate the equation:
   \[
   \frac{1}{g(y)}dy = f(x)dx.
   \]
2. Then integrate:
   \[
   \int \frac{1}{g(y)} \, dy = \int f(x) \, dx.
   \]
3. (If possible) solve for \(y\).

Example 1

\[
\frac{dy}{dx} = (1 + y)e^x, \quad y > -1
\]

- Separate:
  \[
  \frac{1}{1 + y} \, dy = e^x \, dx.
  \]
- Integrate:
  \[
  \int \frac{1}{1 + y} \, dy = \int e^x \, dx,
  \]
  \[
  \ln(1 + y) = e^x + C.
  \]
- Solve \(y\):
  \[
  y(x) = e^{(e^x+C)} - 1.
  \]
Separable differential equations: example 2

\[ \frac{dy}{dx} = -\frac{x}{y} \]

- Separate:
  \[ y \, dy = -x \, dx \]

- Integrate:
  \[ \int y \, dy = -\int x \, dx \]
  \[ \frac{1}{2} y^2 = -\frac{1}{2} x^2 + C \]

- In this case we don not solve \( y \) but write
  \[ x^2 + y^2 = r^2 \]
  where \( r^2 = 2C \).
Exponential change

A system is subject to **exponential change** if it is described by the differential equation

\[
y' = \alpha y
\]

where \(\alpha\) is a constant.

- If \(\alpha > 0\) then we the system is subject to **exponential growth**.
- If \(\alpha < 0\) then we talk about **exponential decay**.
- The differential equation is separable.
**Exponential change**

\[ \frac{dy}{dx} = \alpha y. \]

- Separate:
  \[ \frac{1}{y} \, dy = \alpha \, dx. \]

- Integrate:
  \[ \int \frac{1}{y} \, dy = \alpha \int \, dx, \]
  \[ \ln |y| = \alpha x + C. \]

- Solve y:
  \[ |y| = e^{\alpha x + C} = e^{\alpha x} e^C = M e^{\alpha x} \]

where \( M = e^C \) is a **positive** constant.

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**Exponential change**

\[ \frac{dy}{dx} = \alpha y \quad \Rightarrow \quad |y| = M e^{\alpha x}, \quad M > 0. \]

- Since \( |y| \) is either \( +y \) or \( -y \), we can replace this equation by
  \[ y = L e^{\alpha x}, \]

where \( L = \pm M \) is a non-zero constant.

- Separation fails to find the solution \( y(x) = 0 \), but it certainly is a solution, so if we define \( K = L \) or \( K = 0 \) then
  \[ y(x) = K e^{\alpha x}, \quad K \in \mathbb{R}. \]

- The constant \( K = y(0) \) is the **initial value** of \( y \):
  \[ y(x) = y(0) e^{\alpha x}. \]
### Newton’s Law of Cooling

“*If the temperature of an object is $H(t)$ and the surrounding temperature is $H_S$, then the rate of change in temperature is proportional to the temperature difference $H(t) - H_S$.***

As a differential equation:

$$
\frac{dH}{dt} = -k(H - H_S)
$$

where $k$ is a positive constant.
Newton’s Law of Cooling

\[ \frac{dH}{dt} = -k(H - H_S) \]

- Define \( y(t) = H(t) - H_S \), then
  \[ \frac{dy}{dt} = \frac{d}{dt}(H - H_S) = \frac{dH}{dt} = -k(H - H_S) = -ky. \]

- \( y(t) = y(0)e^{-kt}, \quad K \in \mathbb{R}. \)

- \( H(t) - H_S = (H(0) - H_S)e^{-kt}. \)

- \( H(t) = H_S + (H(0) - H_S)e^{-kt}. \)
Autonomous differential equations

An autonomous differential equation is a differential equation of the form

\[ y' = f(y) \]

- The slope field does not depend on \( x \).
- Along horizontal lines all line segments have the same slope.
- Solution curves can be shifted in horizontal direction.
- If \( y(x) \) is a solution, then \( y(x + C) \) is also a solution.
- If \( x \) represents time:

Solutions of autonomous differential equations are **time independent**.
Example

\[ \frac{dy}{dx} = (y + 1)(y - 2) \]

Above, below and between constant solutions it is sufficient to know whether the slopes are positive or negative.

The phase line

\[ \frac{dy}{dx} = f(y) \]

Solution curves of autonomous differential equations can be sketched qualitatively with just a few computations.

1. Find the equilibrium solutions: solve \( f(y) = 0 \) for \( y \).

2. Draw a phase line:
   - On the \( y \)-axis mark the values for which \( f(y) = 0 \).
   - Identify the intervals where \( f(y) > 0 \) (with an arrow pointing upward: \( \uparrow \)) and \( f(y) < 0 \) (with an arrow pointing downward: \( \downarrow \)).

3. Sketch some solutions in the \( xy \)-plane.

The book also computes the sign of \( y'' \) to determine convexity of solutions. You don’t have to be able to do this!
Example 1

\[ \frac{dy}{dx} = (y + 1)(y - 2) \]

Horizontal phase line

3.5

- Phase lines can also be drawn horizontally.
- If \( f(y) = 0 \), then \( y \) is an equilibrium point.
- If \( f(y) > 0 \), draw an arrow pointing to the right: \( \rightarrow \), and \( f(y) < 0 \) draw an arrow pointing leftward: \( \leftarrow \).
Stability

- An equilibrium is called **asymptotically stable** if the arrows point towards the equilibrium point.
- An equilibrium is called **unstable** if the arrows away from the equilibrium point.
- If $f'(y_0) < 0$, then the equilibrium $y_0$ is stable, and if $f'(y_0) > 0$, then the equilibrium $y_0$ is unstable.

Example 2

$$\frac{dy}{dx} = y^2 - 4$$
Example 2

\[
\frac{dy}{dx} = y^2 - 4 \quad \implies \quad f(y) = y^2 - 4, \quad f'(y) = 2y.
\]

- The equilibrium points are $-2$ and $2$.
- $f$ is positive if and only if $y > 2$ or $y < -2$.
- $f'(-2) = -4 < 0$, hence $-2$ is a stable equilibrium.
- $f'(2) = 4 > 0$, hence $2$ is an unstable equilibrium.

Example 3

\[
\frac{dy}{dx} = y(y - 1)(y + 1)
\]
Example 3

\[
\frac{dy}{dx} = y(y - 1)(y + 1) \quad \implies \quad f(y) = y^3 - y, \quad f'(y) = 3y^2 - 1.
\]

- The equilibrium points are \(-1, 0\) and \(1\).
- \(f\) is positive if and only if \(-1 < y < 0\) or \(y > 1\).
- \(f'(-1) = 3(-1)^2 - 1 = 2 > 0\), hence \(-1\) is unstable.
- \(f'(0) = -1 < 0\), hence \(0\) is stable.
- \(f'(1) = 3 \cdot 1^2 - 1 = 2 > 0\), hence \(1\) is unstable.

What if \(f'(y) = 0\)?

- If the equilibrium point \(y_0\) is a minimum or a maximum of \(f\), then the type of \(y_0\) is **undetermined**.
- If the equilibrium point \(y_0\) is an inflection point and \(f\) is decreasing, then \(y_0\) is asymptotically stable.
- If the equilibrium point \(y_0\) is an inflection point and \(f\) is increasing, then \(y_0\) is unstable.
RC-circuits

- $Q(t)$ is the electric charge in the capacitor.
- Ohm’s law for RC circuits:

$$R \frac{dQ}{dt} + \frac{1}{C} Q = V(t)$$

- If $V(t)$ is constant then the equation is autonomous.
RC-circuits

\[
R \frac{dQ}{dt} + \frac{1}{C} Q = V \quad \Rightarrow \quad \frac{dQ}{dt} = \frac{VC - Q}{RC}
\]

- \( f(Q) = \frac{VC - Q}{RC} \), the equilibrium solution is \( Q_0 = VC \).
- \( f'(Q) = -\frac{1}{RC} < 0 \), the equilibrium \( Q_0 \) is stable.

Skydiving

- On 14th October 2012 Felix Baumgartner set the world record on skydiving at about 39 km.
Skydiving

Newton’s second law of motion states that mass times acceleration equals the sum of all forces acting on Baumgartner.

- The forces acting on Baumgartner are gravity and friction.
  - Gravity acting on an object with mass $m$ is $mg$, where $g \approx 9.81 \text{ m/s}^2$ is the acceleration due to gravity.
  - Assumption: friction is proportional to velocity squared.

- Baumgartner’s downward velocity $v(t)$ satisfies the equation

$$m\dot{v} = mg - kv^2$$

where $\dot{v}$ denotes the derivative of $v$ with respect to time.

The equilibrium solution is $\sqrt{mg/k}$ (the terminal velocity).
A graph of Baumgartner’s actual velocity.

Autonomous differential equations in 1 dimension can only have increasing, decreasing, or constant solutions.

Baumgartner’s real velocity increases and decreases!

Apparently, Baumgartner’s velocity can not be modelled by an autonomous differential equation!

**Plausible explanation:**

- At large altitude there is hardly any air to slow Baumgartner down: friction only acts below a certain altitude.
- The friction constant $k$ changes during the jump!

A more appropriate model for Baumgartner’s velocity also takes the altitude-dependence of friction into account.
Skydiving

■ Altitude follows from downward velocity by integration
\[ h(t) = h(0) - \int_0^t v(s) \, ds \]

■ The **Fundamental Theorem of Calculus** implies
\[ \dot{h} = -v(t) \]

■ A more appropriate model is a **system of two differential equations**:
\[
\begin{align*}
\dot{h} &= -v \\
mv &= mg - k(h)v^2
\end{align*}
\]

where \( k \) is a function of \( h \).

■ Replacing \( v \) by \( -\dot{h} \) in the second equation yields a **second-order differential equation** in \( h \):
\[ -m\ddot{h} = mg - k(h)(\dot{h})^2 \]

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**Exercises**

Assignment: **IMM1 - Tutorial 2.2**